CS357: Second Home Assignment

– First Order Logic–

This assignment is intended to be solved **individually**, but discussion via Piazza is encouraged. Submit your report via email to zeljic@stanford.edu with subject CS357 - Assignment 2. The deadline is **Tuesday November 5$^{\text{th}}$**.

1. Consider the following signature $\Sigma = (D, P)$, with the domain set $D = \{1, 2, 3, 4\}$ and the relation $P = \{(1, 1), (2, 1), (2, 4), (3, 3), (4, 2), (4, 3)\}$. Which of the following Σ-sentences are true:

   (a) $\forall x \, \exists y \, Pxy$

   (b) $\forall x \, (Pxx \vee \exists y \, Pyx)$

   (c) $\exists x \, \forall y \, \neg Pyx$

   (d) $\forall x \, \exists y \, Pxy \wedge Pyx$

2. Suppose P is a binary predicate. Show that no one of the following sentences is logically implied by the other two. Do this by giving a model for each sentence in which the sentence is false but the other two sentences are true.

   (a) $\forall x \, Pxx$

   (b) $\forall x \, \forall y \, (Pxy \vee Pyx \vee x = y)$

   (c) $\exists x \, \forall y \, Pxy$

3. Consider a language with equality and a single binary predicate symbol P. For each set $\mathcal{M}$ of models below, write a first order sentence ϕ such that $\models_M \phi$ iff $M \in \mathcal{M}$.

   (a) $\mathcal{M} = \{M | P^M$ is a transitive relation $\}$.

   (b) $\mathcal{M} = \{M | P^M$ defines a function $\}$.

   (c) $\mathcal{M} = \{M | P^M$ is a bijection (i.e. a function that is 1-1 and onto) $\}$.

4. Consider a signature Σ with no constant symbols, no predicate symbols (except for equality), and a single binary function symbol, $+$. Let M be a Σ-model with domain (the natural numbers) which interprets $+$ in the standard way.

   Note that the only non-logical symbols you may use are $=$ and $+$.

(a) Give a Σ-formula which defines the set $\{0\}$ in M.

(b) Give a Σ-formula which defines the set $\{1\}$ in M.

(c) Give a Σ-formula which defines the binary relation $\{\langle m, n\rangle \,|m < n\}$ in M.

5. Mapping a monotonic function to a sorted array preserves sortedness. Encode this property into AUFLIA logic of SMT-LIB and check whether the property holds.

6. (BONUS) Consider the following puzzle:

   A room has N light switches, numbered by the positive integers 1 through N. There are also N children, numbered by the positive integers 1 through N. Initially, the switches are all off. Each child k enters the room and changes the position of every light switch n such that n is a multiple of k. That is, child 1 changes all the switches, child 2 changes switches 2, 4, 6, 8, ..., child 3 changes switches 3, 6, 9, 12, ..., etc., and child N changes only light switch N. When all the children have gone through the room, how many of the light switches are on?

   1 Write a program that encodes this puzzle for a given value of N in SMT-LIB format using the logic of quantifier-free bit-vectors. For the report, describe what kinds of constraints are generated.

   2 Download and install CVC4 from `cvc4.github.io`. What is the largest N CVC4 can solve the problem for, within 10minutes.

   3 Intuitively, what is the solution to the puzzle?

For the SMT-LIB problems the documentation can be found at `http://smtlib.cs.uiowa.edu`. It can be useful to use online interfaces to SMT-solvers to work through some examples — CVC4 interface and Z3 interface.