Satisfiability Modulo Theories

Materials by Clark Barrett, Stanford University

CS357: October 2019

Acknowledgments: Many thanks to Cesare Tinelli and Albert Oliveras for contributing some of the material used in these slides.

Disclamer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

Introduction

The Satisfiability Revolution

Princeton, c. 2000

- *Chaff SAT solver*: orders of magnitude faster than previous SAT solvers
- *Important observation*: many real-world problems do not exhibit worst-case theoretical performance

Palo Alto, c. 2001

- Idea: combine fast new SAT solvers with decision procedures for decidable first-order theories
- *SVC*, *CVC* solvers (Stanford); *ICS*, *Yices* solvers (SRI)
- *Satisfiability Modulo Theories* (SMT) was born

SMT solvers

SMT solvers: *general-purpose* logic engines

- Given condition X, is it possible for Y to happen
- X and Y are expressed in a *rich logical language*
  - First-order logic
  - Domain-specific reasoning
    - arithmetic, arrays, bit-vectors, data types, etc.

SMT solvers are *changing the way people solve problems*

- Instead of building a *special-purpose* solver
- *Translate* into a logical formula and use an SMT solver
- Not only easier, *often better*

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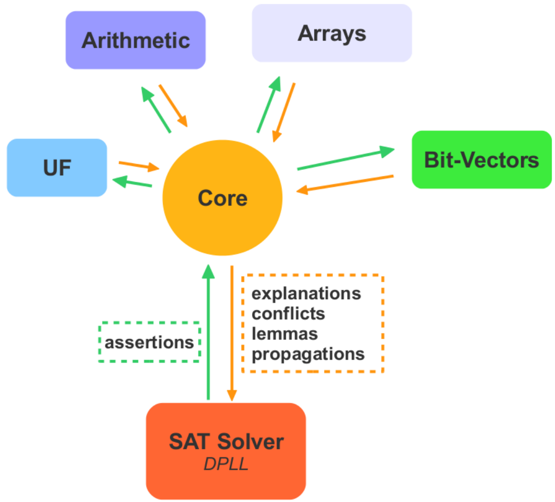
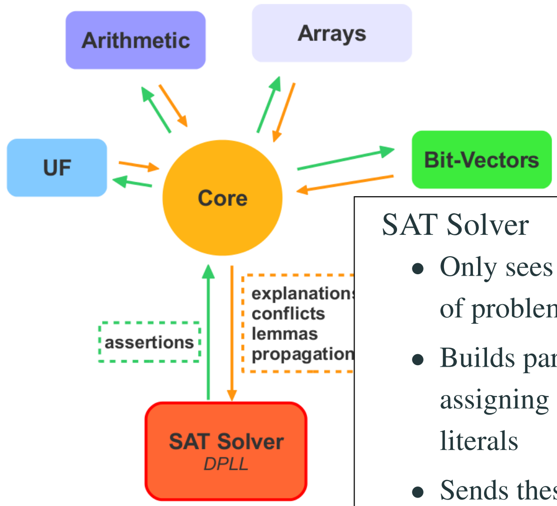

Arithmetic
Arrays
UF
Core
Bit-Vectors
assertions
explanations
conflicts
lemmas
propagations
SAT Solver
DPLL

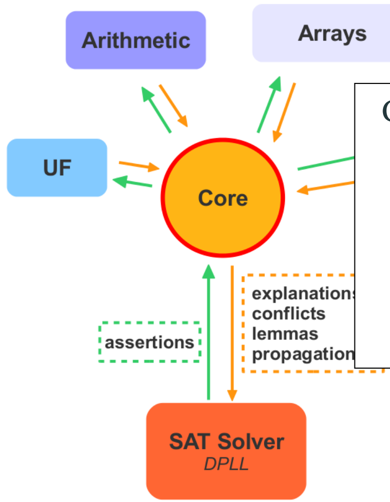
SMT Solvers



SAT Solver

- Only sees *Boolean skeleton* of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as *assertions*

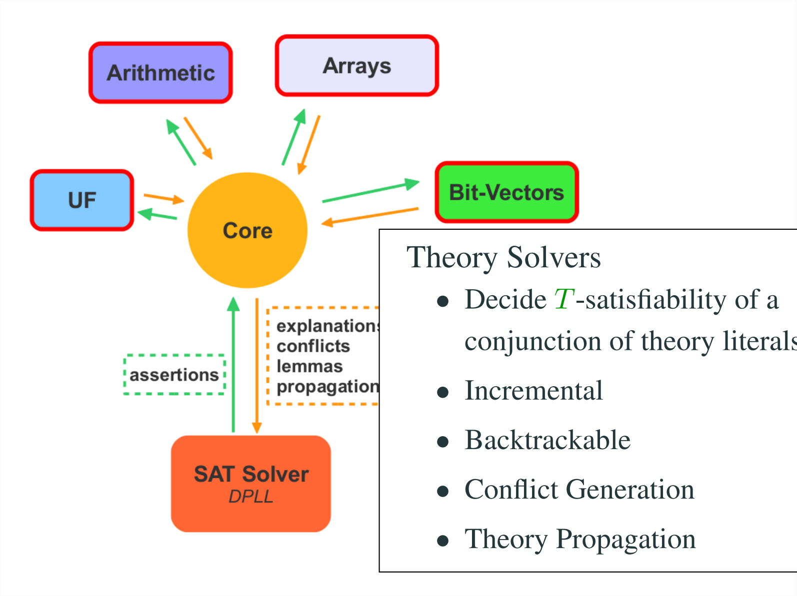
SMT Solvers



Core

- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

SMT Solvers



Theory Solvers

- Decide T-satisfiability of a conjunction of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation

DPLL(T): Combining T-Solvers with SAT

Satisfiability Modulo a Theory T

Def. A formula is *(un)satisfiable in* a theory T, or T*-(un)satisfiable*, if there is a (no) model of T that satisfies it

Note: The T-satisfiability of quantifier-free formulas is decidable iff the T-satisfiability of conjunctions/sets of literals is decidable

(Convert the formula in DNF and check if any of its disjuncts is T-sat)

Problem: In practice, dealing with Boolean combinations of literals is as hard as in propositional logic

Solution: Exploit propositional satisfiability technology

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Lifting SAT Technology to SMT

Two main approaches:

1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]
   - translate into an equisatisfiable propositional formula
   - feed it to any SAT solver

Notable systems: *UCLID*

2. "Lazy" [ACG00, dMR02, BDS02, ABC+02]
   - abstract the input formula to a propositional one
   - feed it to a (DPLL-based) SAT solver
   - use a theory decision procedure to refine the formula and guide the SAT solver

Notable systems: *Barcelogic*, *Boolector*, *CVC4*, *MathSAT*, *Yices*, *Z3*

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(Very) Lazy Approach for SMT – Example

$$g(a) = c \quad \wedge \quad f(g(a)) \neq f(c) \; \vee \; g(a) = d \quad \wedge \quad c \neq d$$

Theory T**:** Equality with Uninterpreted Functions

Simplest setting:

- Off-line SAT solver
- Non-incremental *theory solver* for conjunctions of equalities and disequalities
- Theory atoms (e.g., $g(a) = c$) abstracted to propositional atoms (e.g., 1)

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- Send $\{1,\ \overline{2} \vee 3,\ \overline{4}\}$ to SAT solver.
- SAT solver returns model $\{1,\ \overline{2},\ \overline{4}\}$.
  Theory solver finds (concretization of) $\{1,\ \overline{2},\ \overline{4}\}$ unsat.
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  Done: the original formula is unsatisfiable in UF.

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Lazy Approach – Enhancements

Several enhancements are possible to increase efficiency:

- Check T-satisfiability only of full propositional model
- Check T-satisfiability of partial assignment M as it grows
-
- If M is T-unsatisfiable, identify a T-unsatisfiable subset M_0 of M and add $\neg M_0$ as a clause
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Lazy Approach – Main Benefits

- Every tool does what it is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
  - SAT and theory solvers communicate via a simple API [GHN+04]
  - SMT for a new theory only requires new theory solver
  - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)

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An Abstract Framework for Lazy SMT

Several variants and enhancements of lazy SMT solvers exist

They can be modeled abstractly and declaratively as *transition systems*

A transition system is a binary relation over states, induced by a set of conditional transition rules

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07]

Advantages of Abstract Framework

An abstract framework helps one:

- skip over implementation details and unimportant control aspects
- reason formally about solvers for SAT and SMT
- model advanced features such as non-chronological bactracking, lemma learning, theory propagation, ...
- describe different strategies and prove their correctness
- compare different systems at a higher level
- get new insights for further enhancements

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The Original DPLL Procedure

- Modern SAT solvers are based on the DPLL procedure [DP60, DLL62]
- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F
- M is grown by
  - deducing the truth value of a literal from M and F, or
  - guessing a truth value
- If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

An Abstract Framework for DPLL

States:

$$\mathsf{fail} \quad \text{or} \quad \langle M, F \rangle$$

where

- M is a sequence of literals and *decision points* $\bullet$ denoting a partial truth *assignment*
- F is a set of clauses denoting a CNF *formula*

Def. If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level* i of M
- $M^{[i]} \stackrel{\text{def}}{=} M_0 \bullet \cdots \bullet M_i$

An Abstract Framework for DPLL

States:

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Initial state:

- $\langle (), F_0 \rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- $\textsf{fail}$ if F_0 is unsatisfiable
- $\langle M, G \rangle$ otherwise, where
    - G is equivalent to F_0 and
    - M satisfies G

Transition Rules: Notation

States treated like records:

- M denotes the truth assignment component of current state
- F denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

$$\frac{p_1 \quad \cdots \quad p_n}{[\mathsf{M} := e_1] \quad [\mathsf{F} := e_2]}$$

updating M, F or both when premises $p_1, \ldots, p_n$ all hold

Transition Rules for the Original DPLL

Extending the assignment

Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

Note: When convenient, treat M as a set

Decide $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

Note: $\mathrm{Lit}(F) \stackrel{\text{def}}{=} \{l \mid l \text{ literal of } F\} \cup \{\bar{l} \mid l \text{ literal of } F\}$

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Transition Rules for the Original DPLL

Repairing the assignment

Fail $\dfrac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \bullet \notin \mathsf{M}}{\mathsf{fail}}$

Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\, N \quad \bullet \notin N}{\mathsf{M} := M\, \bar{l}}$$

Note: Last premise of **Backtrack** enforces chronological backtracking

Transition Rules for the Original DPLL

Repairing the assignment

Fail $$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\, N \quad \bullet \notin N}{\mathsf{M} := M\, \bar{l}}$$

Note: Last premise of **Backtrack** enforces chronological backtracking

From DPLL to CDCL Solvers (1)

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

States: fail or $\langle M, F, C \rangle$

Initial state:

- $\langle (), F_0, \mathsf{no} \rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- fail if F_0 is unsatisfiable
- $\langle M, G, \mathsf{no} \rangle$ otherwise, where
  - G is equivalent to F_0 and
  - M satisfies G

From DPLL to CDCL Solvers (1)

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

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  - M satisfies G

From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

$$\textbf{Conflict}\ \frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

$$\textbf{Explain}\ \frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

$$\textbf{Backjump}\ \frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\ \bar{l}_1, \ldots, \mathsf{lev}\ \bar{l}_n \leq\ i < \mathsf{lev}\ \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\ l}$$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Note: $\models_{\mathrm{p}}$ denotes propositional entailment

From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

Explain $$\frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Backjump $$\frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\, \bar{l}_1, \ldots, \mathsf{lev}\, \bar{l}_n \leq\ i < \mathsf{lev}\, \bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\, l}$$

Note: $l \prec_{\mathsf{M}} l'$ if l occurs before l' in M
$\mathsf{lev}\, l = i$ iff l occurs in decision level i of M

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Note: $\models_{\mathrm{p}}$ denotes propositional entailment

From DPLL to CDCL Solvers (2)

Replace **Backtrack** with

$$\textbf{Conflict}\ \frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

$$\textbf{Explain}\ \frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

$$\textbf{Backjump}\ \frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\,\bar{l}_1, \ldots, \mathsf{lev}\,\bar{l}_n \leq\ i < \mathsf{lev}\,\bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\, l}$$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Note: $\models_{\mathrm{p}}$ denotes propositional entailment

Modify **Fail** to

Fail $\dfrac{C \neq \mathsf{no} \quad \bullet \notin M}{\mathsf{fail}}$

From DPLL to CDCL Solvers (3)

Modify **Fail** to

$$\textbf{Fail}\ \frac{\mathsf{C} \neq \mathsf{no} \qquad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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	F	no	
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$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{1} \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$F := \{1,\ \overline{1}\vee 2,\ \overline{3}\vee 4,\ \overline{5}\vee\overline{6},\ \overline{1}\vee\overline{5}\vee 7,\ \overline{2}\vee\overline{5}\vee 6\vee\overline{7}\}$

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$1\ 2\bullet 3\ 4\bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2\bullet 3\ 4\bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2\bullet 3\ 4\bullet 5\ \overline{6}\ 7$	F	$\overline{2}\vee\overline{5}\vee 6\vee\overline{7}$	by **Conflict**
$1\ 2\bullet 3\ 4\bullet 5\ \overline{6}\ 7$	F	$1\vee\overline{2}\vee\overline{5}\vee 6$	by **Explain** with $\overline{1}\vee\overline{5}\vee 7$
$1\ 2\bullet 3\ 4\bullet 5\ \overline{6}\ 7$	F	$1\vee\overline{2}\vee\overline{5}$	by **Explain** with $\overline{5}\vee\overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5}\bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \lor 2,\ \overline{3} \lor 4,\ \overline{5} \lor \overline{6},\ \overline{1} \lor \overline{5} \lor 7,\ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \lor \overline{5} \lor 6 \lor \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5} \lor 6$	by **Explain** with $\overline{1} \lor \overline{5} \lor 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $5 \vee 6$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
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1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Execution Example

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
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$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

Execution Example

$$F := \{1,\ \overline{1} \lor 2,\ \overline{3} \lor 4,\ \overline{5} \lor \overline{6},\ \overline{1} \lor \overline{5} \lor 7,\ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

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	F	no	
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$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \lor \overline{5} \lor 6 \lor \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5} \lor 6$	by **Explain** with $\overline{1} \lor \overline{5} \lor 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\dots$			

From DPLL to CDCL Solvers (4)

Also add

Learn $$\frac{\mathsf{F} \models_{\mathrm{p}} C \quad C \notin \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{C\}}$$

Forget $$\frac{\mathsf{C} = \mathsf{no} \quad \mathsf{F} = G \cup \{C\} \quad G \models_{\mathrm{p}} C}{\mathsf{F} := G}$$

Restart $$\frac{}{\mathsf{M} := \mathsf{M}^{[0]} \quad \mathsf{C} := \mathsf{no}}$$

Note: Learn can be applied to any clause stored in C when $\mathsf{C} \neq \mathsf{no}$

Modeling Modern SAT Solvers

At the core, current CDCL SAT solvers are implementations of the transition system with rules

Propagate, **Decide**,

Conflict, **Explain**, **Backjump**,

Learn, **Forget**, **Restart**

Basic DPLL $\stackrel{\text{def}}{=}$

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DPLL $\stackrel{\text{def}}{=}$ Basic DPLL + { **Learn**, **Forget**, **Restart** }

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The Basic DPLL System – Correctness

Some terminology:

Irreducible state: state for which no Basic DPLL rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, the clause set F_0 is satisfied by M.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Note: This is not so immediate, because of **Backjump**.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

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The DPLL System – Strategies

- Applying
  - one Basic DPLL rule between each two **Learn** applications and
  - **Restart** less and less often

  ensures termination

- A common basic strategy applies the rules with the following priorities:
  1. If $n > 0$ conflicts have been found so far, increase n and apply **Restart**
  2. If a clause is falsified by M, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
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  6. Apply **Propagate** to completion
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From SAT to SMT

Same states and transitions but

- F contains quantifier-free clauses in some theory T
- M is a sequence of theory literals and decision points
- the DPLL system is augmented with rules

  T**-Conflict**, T**-Propagate**, T**-Explain**

- maintains invariant: $\mathsf{F} \models_T \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Def. $F \models_T G$ iff every model of T that satisfies F satisfies G as well

SMT-level Rules

Fix a theory T

T**-Conflict** $$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_T \bot}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T**-Propagate** $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T**-Explain** $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

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Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

Modeling the Very Lazy Theory Approach

T-**Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier UF example

$$\underbrace{g(a)=c}_{1} \quad\wedge\quad \underbrace{f(g(a))\neq f(c)}_{\overline{2}} \vee \underbrace{g(a)=d}_{3} \quad\wedge\quad \underbrace{c\neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4}$	$\overline{1}\vee 2\vee 4$	by *T*-**Conflict**
$1\ \overline{4}\bullet\overline{2}$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	$\overline{1}\vee 2\vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2}\vee 3,\ \overline{4},\ \overline{1}\vee 2\vee 4,\ \overline{1}\vee\overline{3}\vee 4$	$\overline{1}\vee\overline{3}\vee 4$	by *T*-**Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \lor \underbrace{g(a) = d}_{3} \quad \land \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Propagate$^+$**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	$\overline{1} \lor 2 \lor 4$	by ***T*-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4},\ \overline{1} \lor 2 \lor 4$	$\overline{1} \lor 2 \lor 4$	by **Learn**
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fail			by **Fail**

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
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fail			by **Fail**

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

Modeling the Very Lazy Theory Approach

$$\underbrace{g(a) = c}_{1} \;\wedge\; \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \;\wedge\; \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by **T-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by **T-Conflict**, **Learn**
fail			by **Fail**

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by T**-Conflict**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by T**-Conflict**, **Learn**
fail			by **Fail**

A Better Lazy Approach

The very lazy approach can be improved considerably with

- An *on-line* SAT engine,
  which can accept new input clauses on the fly
- an *incremental and explicating* T-solver,
  which can
  1. check the T-satisfiability of M as it is extended and
  2. identify a small T-unsatisfiable subset of M once M becomes T-unsatisfiable

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$$\underbrace{g(a) = c}_{1} \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \lor \underbrace{g(a) = d}_{3} \quad \land \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Decide**
$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \lor 3,\ \overline{4}$	$\overline{1} \lor 2$	by **T-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \lor 3,\ \overline{4}$	$\overline{1} \lor \overline{3} \lor 4$	by **T-Conflict**
fail			by **Fail**

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	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
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$1\ \overline{4}\ \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ \overline{2}\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ \overline{2}\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by $\mathcal{T}$**-Conflict**
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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**
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fail			by **Fail**

Lazy Approach – Strategies

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
2. If M is T-unsatisfiable, apply T**-Conflict**
3. Apply **Fail** or **Explain+Learn+Backjump** as appropriate
4. Apply **Propagate**
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Note: Depending on the cost of checking the T-satisfiability of M, Step (2) can be applied with lower frequency or priority

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Theory Propagation

With T**-Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

With T**-Propagate** and T**-Explain**, it can also be used to guide the engine's search [Tin02]

T**-Propagate** $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

T**-Explain** $$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

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$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T**-Propagate** $(1 \models_T 2)$
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fail			by **Fail**

Note: T-propagation eliminates search altogether in this case
no applications of **Decide** are needed

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$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^+$
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** $(1 \models_T 2)$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** $(1,\ \overline{4} \models_T \overline{3})$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{2} \vee 3$	by **Conflict**
fail			by **Fail**

Note: T-propagation eliminates search altogether in this case
no applications of **Decide** are needed

Modeling Modern Lazy SMT Solvers

At the core, current lazy SMT solvers are implementations of the transition system with rules

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) T**-Conflict**, T**-Propagate**, T**-Explain**

(3) **Learn**, **Forget**, **Restart**

Basic DPLL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2)

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Correctness

Updated terminology:

Irreducible state: state to which no Basic DPLL MT rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is T-unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, F_0 is T-satisfiable; specifically, M is T-satisfiable and $\mathsf{M} \models_{\mathrm{p}} F_0$.

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Proposition (Termination) Every execution in which

(a) **Learn/Forget** are applied only finitely many times and

(b) **Restart** is applied with increased periodicity

is finite.

Lemma Every exhausted execution ends with either $\mathsf{C} = \mathsf{no}$ or fail.

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DPLL(T) Architecture

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DPLL(T) = DPLL(X) engine + T-solver

DPLL(X):

- Very similar to a SAT solver, enumerates Boolean models
- Not allowed: pure literal, blocked literal detection, ...
- Required: incremental addition of clauses
- Desirable: partial model detection

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

T-solver:

- Checks the T-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of T-unsatisfiability/propagation
- Must be incremental and backtrackable

Reasoning by Cases in Theory Solvers

For certain theories, determining that a set M is T-unsatisfiable requires reasoning by cases.

Example: T = the theory of arrays.

$$M = \{\underbrace{r(w(a,i,x),j) \neq x}_{1},\ \underbrace{r(w(a,i,x),j) \neq r(a,j)}_{2}\}$$

$i = j$) Then, $r(w(a,i,x),j) = x$. Contradiction with 1.

$i \neq j$) Then, $r(w(a,i,x),j) = r(a,j)$. Contradiction with 2.

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Case Splitting

A *complete* T-solver reasons by cases via (internal) case splitting and backtracking mechanisms

An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine

Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them [BNOT06]

Possible benefits:

- All case-splitting is coordinated by the SAT engine
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Splitting on Demand

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Basic Scenario:

$$\mathsf{M} = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

- Main SMT module: "Is M $\mathcal{T}$-unsatisfiable?"
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$$s = s' \wedge i = j \rightarrow s = t, \quad s = s' \wedge i \neq j \rightarrow s = r(a,j)\text{"}$$

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To model the generation of theory lemmas for case splits, add the rule

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$$\frac{\models_T \exists \mathbf{v}(l_1 \vee \cdots \vee l_n) \quad l_1, \ldots, l_n \in L_{\mathrm{S}} \quad \mathbf{v} \text{ vars not in } \mathsf{F}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

Note: For many theories with a theory solver, there exists an appropriate finite L_{S} for every input F

The set L_{S} does not need to be computed explicitly

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Now we can relax the requirement on the theory solver:

When $\mathsf{M} \models_{\mathrm{p}} \mathsf{F}$*, it must either*

- *determine whether* $\mathsf{M} \models_T \bot$ *or*
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Example — Theory of Finite Sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \ \bullet \ y = \emptyset$	F	by **Decide**
$x = y \cup z \ \bullet \ y = \emptyset \ x \neq z$	F	by **Propagate**
$x = y \cup z \ \bullet \ y = \emptyset \ x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by $\mathcal{T}$**-Learn**
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T-solver can make the following deductions at this point:

$$e \in x \quad \cdots \quad \Rightarrow \quad e \in y \cup z \quad \cdots \quad \Rightarrow \quad e \in y \quad \cdots \quad \Rightarrow \quad e \in \emptyset \quad \Rightarrow \quad \bot$$

This enables an application of T-**Conflict** with clause

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M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \;\bullet\; y = \emptyset$	F	by **Decide**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z$	F	by **Propagate**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z$	$F, (x = z \vee e \in x \vee e \in z), (x = z \vee e \notin x \vee e \notin z)$	by T**-Learn**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z \;\bullet\; e \in x$	$F, (x = z \vee e \in x \vee e \in z), (x = z \vee e \notin x \vee e \notin z)$	by **Decide**
$x = y \cup z \;\bullet\; y = \emptyset\; x \neq z \;\bullet\; e \in x\; e \notin z$	$F, (x = z \vee e \in x \vee e \in z), (x = z \vee e \notin x \vee e \notin z)$	by **Propagate**

T-solver can make the following deductions at this point:

$$e \in x \;\; \cdots \;\; \Rightarrow \;\; e \in y \cup z \;\; \cdots \;\; \Rightarrow \;\; e \in y \;\; \cdots \;\; \Rightarrow \;\; e \in \emptyset \;\; \Rightarrow \;\; \bot$$

This enables an application of T**-Conflict** with clause

$$x \neq y \cup z \;\vee\; y \neq \emptyset \;\vee\; x = z \;\vee\; e \notin x \;\vee\; e \in z$$

Example — Theory of Finite Sets

$$F: \ x = y \cup z \ \wedge \ y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \bullet y = \emptyset$	F	by **Decide**
$x = y \cup z \bullet y = \emptyset \; x \neq z$	F	by **Propagate**
$x = y \cup z \bullet y = \emptyset \; x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **T-Learn**
$x = y \cup z \bullet y = \emptyset \; x \neq z \bullet e \in x$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Decide**
$x = y \cup z \bullet y = \emptyset \; x \neq z \bullet e \in x \; e \notin z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by **Propagate**

T-solver can make the following deductions at this point:

$$e \in x \ \cdots \ \Rightarrow \ e \in y \cup z \ \cdots \ \Rightarrow \ e \in y \ \cdots \ \Rightarrow \ e \in \emptyset \ \Rightarrow \ \bot$$

This enables an application of **T-Conflict** with clause

$$x \neq y \cup z \ \vee \ y \neq \emptyset \ \vee \ x = z \ \vee \ e \notin x \ \vee \ e \in z$$

Example — Theory of Finite Sets

$$F: \;\; x = y \cup z \;\; \wedge \;\; y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by **Propagate**$^+$
$x = y \cup z \; \bullet \; y = \emptyset$	F	by **Decide**
$x = y \cup z \; \bullet \; y = \emptyset \; x \neq z$	F	by **Propagate**
$x = y \cup z \; \bullet \; y = \emptyset \; x \neq z$	$F, (x = z \vee e \in x \vee e \in z), (x = z \vee e \notin x \vee e \notin z)$	by T**-Learn**
$x = y \cup z \; \bullet \; y = \emptyset \; x \neq z \; \bullet \; e \in x$	$F, (x = z \vee e \in x \vee e \in z), (x = z \vee e \notin x \vee e \notin z)$	by **Decide**
$x = y \cup z \; \bullet \; y = \emptyset \; x \neq z \; \bullet \; e \in x \; e \notin z$	$F, (x = z \vee e \in x \vee e \in z), (x = z \vee e \notin x \vee e \notin z)$	by **Propagate**

T-solver can make the following deductions at this point:

$$e \in x \;\; \cdots \;\; \Rightarrow \;\; e \in y \cup z \;\; \cdots \;\; \Rightarrow \;\; e \in y \;\; \cdots \;\; \Rightarrow \;\; e \in \emptyset \;\; \Rightarrow \;\; \bot$$

This enables an application of T**-Conflict** with clause

$$x \neq y \cup z \;\vee\; y \neq \emptyset \;\vee\; x = z \;\vee\; e \notin x \;\vee\; e \in z$$

Correctness Results

Correctness results can be extended to the new rule.

Soundness: The new T-**Learn** rule maintains satisfiability of the clause set.

Completeness: As long as the theory solver can decide $\mathsf{M} \models_T \bot$ when all literals in L_{S} are determined, the system is still complete.

Termination: The system terminates under the same conditions as before. Roughly:

- Any lemma is (re)learned only finitely many times
- **Restart** is applied with increased periodicity

Suggested Readings

1. R. Nieuwenhuis, A. Oliveras, and C. Tinelli. **Solving SAT and SAT Modulo Theories: From an abstract Davis-Putnam-Logemann- Loveland procedure to DPLL(T)**. Journal of the ACM, 53(6):937-977, 2006.

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