Nelson-Open Theory Combination

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Disclamer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

Combining Theories

Need for Combining Theories and Solvers

Recall: Many applications give rise to formulas like:

$$a \approx b + 2 \,\wedge\, A \approx \mathrm{write}(B, a+1, 4) \,\wedge$$
$$(\mathrm{read}(A, b+3) \approx 2 \,\vee\, f(a-1) \neq f(b+1))$$

Solving that formula requires reasoning over

- the theory of linear arithmetic (T_{LA})
- the theory of arrays (T_{A})
- the theory of uninterpreted functions (T_{UF})

Question: Given solvers for each theory, can we combine them modularly into one for $T_{\mathrm{LA}} \cup T_{\mathrm{A}} \cup T_{\mathrm{UF}}$?

Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]

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Motivating Example (Convex Case)

Consider the following set of literals over $T_{\mathrm{LRA}} \cup T_{\mathrm{UF}}$ (T_{LRA}, linear real arithmetic):

$$\begin{aligned} f(f(x) - f(y)) &= a \\ f(0) &> a + 2 \\ x &= y \end{aligned}$$

First step: *purify* literals so that each belongs to a single theory

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$$f(f(x) - f(y)) = a \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = f(x) - f(y) \end{array} \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = e_2 - e_3 \\ e_2 = f(x) \\ e_3 = f(y) \end{array}$$

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$$f(0) > a + 2 \quad \Longrightarrow \quad \begin{array}{c} f(e_4) > a + 2 \\ e_4 = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{c} f(e_4) = e_5 \\ e_4 = 0 \\ e_5 > a + 2 \end{array}$$

Motivating Example (Convex Case)

Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

L_1	L_2
$f(e_1) = a$	$e_2 - e_3 = e_1$
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$f(y) = e_3$	$e_5 > a + 2$
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$L_1 \models_{\mathrm{UF}} e_2 = e_3 \qquad L_2 \models_{\mathrm{LRA}} e_1 = e_4$

$L_1 \models_{\mathrm{UF}} a = e_5$

Third step: check for satisfiability locally

Report unsatisfiable

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Motivating Example (Non-convex Case)

Consider the following unsatisfiable set of literals over $T_{\text{LIA}} \cup T_{\text{UF}}$ (T_{LIA}, linear integer arithmetic):

$$\begin{aligned} 1 \leq \quad x & \quad \leq 2 \\ f(1) &= a \\ f(2) &= f(1) + 3 \\ a &= b + 2 \end{aligned}$$

First step: *purify* literals so that each belongs to a single theory

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$$f(1) = a \quad \Longrightarrow \quad \begin{array}{r} f(e_1) = a \\ e_1 = 1 \end{array}$$

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$$f(2) = f(1) + 3 \quad \Longrightarrow \quad \begin{aligned} e_2 &= 2 \\ f(e_2) &= e_3 \\ f(e_1) &= e_4 \\ e_3 &= e_4 + 3 \end{aligned}$$

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No more entailed equalities, but $L_1 \models_{\text{LIA}} x = e_1 \lor x = e_2$

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Consider each case of $x = e_1 \vee x = e_2$ separately

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Case 1) $x = e_1$

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$L_2 \models_{\mathrm{UF}} a = b$, which entails $\perp$ when sent to L_1

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Case 2) $x = e_2$

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$L_2 \models_{\mathrm{UF}} e_3 = b$, which entails $\bot$ when sent to L_1

The Nelson-Oppen Method

- For $i = 1, 2$, let T_i be a first-order theory of *signature* Σ_i (set of function and predicate symbols in T_i other than $=$)
- Let $T = T_1 \cup T_2$
- Let $\mathcal{C}$ be a finite set of *free* constants (i.e., not in $\Sigma_1 \cup \Sigma_2$)

We consider only input problems of the form

$$L_1 \cup L_2$$

where each L_i is a finite set of *ground* (i.e., variable-free) $(\Sigma_i \cup \mathcal{C})$-literals

Note: Because of purification, there is no loss of generality in considering only ground $(\Sigma_i \cup \mathcal{C})$-literals

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Bare-bones, non-deterministic, non-incremental version

[Opp80, Rin96, TH96]:

Input: $L_1 \cup L_2$ with L_i finite set of ground $(\Sigma_i \cup \mathcal{C})$-literals
Output: **sat** or **unsat**

1. Guess an *arrangement* A, i.e., a set of equalities and disequalities over $\mathcal{C}$ such that

$$c = d \in A \text{ or } c \neq d \in A \text{ for all } c, d \in \mathcal{C}$$

2. If $L_i \cup A$ is T_i-unsatisfiable for $i = 1$ or $i = 2$, return unsat
3. Otherwise, return **sat**

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2. If $L_i \cup A$ is T_i-unsatisfiable for $i = 1$ or $i = 2$, return unsat

3. Otherwise, return **sat**

The Nelson-Oppen Method

Bare-bones, non-deterministic, non-incremental version

[Opp80, Rin96, TH96]:

Input: $L_1 \cup L_2$ with L_i finite set of ground $(\Sigma_i \cup \mathcal{C})$-literals
Output: **sat** or **unsat**

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Correctness of the NO Method

Proposition (Termination) The method is terminating.

(Trivially, because there is only a finite number of arrangements to guess)

Proposition (Soundness) If the method returns **unsat** for every arrangement, the input is $(T_1 \cup T_2)$-unsatisfiable.

(Because satisfiability in $(T_1 \cup T_2)$ is always preserved)

Proposition (Completeness) If $\Sigma_1 \cap \Sigma_2 = \emptyset$ and T_1 and T_2 are stably infinite, when the method returns **sat** for some arrangement, the input is $(T_1 \cup T_2)$-is satisfiable.

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Stably Infinite Theories

Def. A theory T is *stably infinite* iff every quantifier-free T-satisfiable formula is satisfiable in an infinite model of T

Many interesting theories are stably infinite:

- Theories of an infinite structure (e.g., integer arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
- Convex theories (e.g., EUF, linear real arithmetic)

Def. A theory T is *convex* iff, for any set L of literals
$L \models_T s_1 = t_1 \vee \cdots \vee s_n = t_n \implies L \models_T s_i = t_i$ for some i

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SMT Solving with *Multiple* Theories

Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$

How can we integrate all of them cooperatively into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

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Quick Solution:

1. Combine $S_1, \ldots, S_n$ with Nelson-Oppen into a theory solver for T
2. Build a DPLL(T) solver as usual

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Better Solution [Bar02, BBC+05b, BNOT06]:

1. Extend DPLL(T) to DPLL($T_1, \ldots, T_n$)
2. Lift Nelson-Oppen to the DPLL($X_1, \ldots, X_n$) level
3. Build a DPLL($T_1, \ldots, T_n$) solver

Modeling DPLL($T_1, \ldots, T_n$) Abstractly

- Let $n = 2$, for simplicity
- Let T_i be of signature Σ_i for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$
- Let $\mathcal{C}$ be a set of free constants
- Assume wlog that each input literal has signature $(\Sigma_1 \cup \mathcal{C})$ or $(\Sigma_2 \cup \mathcal{C})$ (no *mixed* literals)
- Let $\mathsf{M}|_i \stackrel{\text{def}}{=} \{(\Sigma_i \cup \mathcal{C})\text{-literals of } \mathsf{M} \text{ and their complement}\}$
- Let $\mathrm{I}(\mathsf{M}) \stackrel{\text{def}}{=} \{c = d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\} \cup$
  $\{c \neq d \mid c, d \text{ occur in } \mathcal{C}, \mathsf{M}|_1 \text{ and } \mathsf{M}|_2\}$

  (*interface literals*)

Abstract DPLL Modulo Multiple Theories

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

Decide $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$

Only change: decide on interface equalities as well

T-Propagate $\dfrac{l \in \mathrm{Lit}(\mathsf{F}) \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\} \quad \mathsf{M} \models_{T_i} l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$

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Abstract DPLL Modulo Multiple Theories

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each T_i

I-Learn

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathrm{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

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Example — Convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$

$$\underbrace{e_2 - e_3 = e_1}_{5} \wedge \underbrace{e_4 = 0}_{6} \wedge \underbrace{e_5 > a + 2}_{7}$$

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M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** $(1, 2, 4 \models_{\text{UF}} 8)$
0 1 2 3 4 5 6 7 8 9	F	no	by T-**Propagate** $(5, 6, 8 \models_{\text{LRA}} 9)$
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0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \vee \overline{10}$	by T-**Conflict** $(7, 10 \models_{\text{LRA}} \bot)$
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F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge
$$

$$
\underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}
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$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

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$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ 3 \models_{\text{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\text{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\text{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\text{UF}} \perp)$
$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\text{UF}} \overline{11})$
$0 \cdots 9\ 10\ 13\ 11\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
$$\underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ 13\ 11\ 12$	$F,\ 4 \vee 5 \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ \underline{3} \models_{UF} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{LIA} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{UF} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{UF} \bot)$
$0 \cdots 9\ \underline{10}\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{13}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ 13\ 11\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \begin{array}{l} \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge \\ \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9} \end{array}$$

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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ \overline{3} \models_{\text{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\text{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\text{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\text{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\text{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^{+}$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ \underline{10}\ \underline{\overline{13}}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \overline{\underline{10}}\ \overline{\underline{13}}\ 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ 13\ 11\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = b}^{1} \land \overbrace{f(e_2) = e_3}^{2} \land \overbrace{f(e_1) = e_4}^{3} \land$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{UF} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **I-Learn** $(\models_{LIA} \overline{4} \lor \overline{5} \lor 11 \lor 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{UF} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	$\overline{7} \lor \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{UF} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **T-Propagate** $(0,\ 1,\ \overline{13} \models_{UF} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
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Example — Non-convex Theories

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge \underbrace{1 \le x}_{4} \wedge \underbrace{x \le 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

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M	F	C	rule
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$0 \cdots 9$	F	no	by **Propagate**$^{+}$
$0 \cdots 9\ 10$	F	no	by T**-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
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$$F := \begin{array}{l} \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge \\ \underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9} \end{array}$$

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$0 \cdots 9$	F	no	by **Propagate**$^+$
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$0 \cdots 9\ 10$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ \underline{13}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T**-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ \underline{10}\ \underline{\overline{13}}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ \underline{10}\ \underline{\overline{13}}\ \overline{11}$	$F,\ \underline{\overline{4}} \vee \underline{\overline{5}} \vee 11 \vee 12$	no	by T**-Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
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$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\ldots$			(exercise)
fail	$\ldots$	$\ldots$	by **Fail**

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$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
$$\underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^{+}$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by I-**Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by T-**Conflict** $(7,\ 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by T-**Propagate** $(0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11})$
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\dots$			(exercise)
fail	$\dots$	$\dots$	by **Fail**

Theory Solvers

Theory Solvers

Given a theory T, a *Theory Solver* for T takes as input a set Φ of literals and determines whether Φ is T-satisfiable.

Φ is T-satisfiable iff there is some model M of T such that each formula in Φ holds in M.

Theories of Interest: UF

Equality ($=$) with Uninterpreted Functions [NO80, BD94, NO07]

Typically used to abstract unsupported constructs, e.g.:

- non-linear multiplication in arithmetic
- ALUs in circuits

Example: The formula

$$a * (|b| + c) = d \quad \wedge \quad b * (|a| + c) \neq d \quad \wedge \quad a = b$$

is unsatisfiable, but no arithmetic reasoning is needed

if we abstract it to

$$mul(a, add(abs(b), c)) = d \quad \wedge \quad mul(b, add(abs(a), c)) \neq d \quad \wedge \quad a = b$$

Theories of Interest: Arithmetic

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$ [BBC+05a]
- Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
- Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06]
- Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM+09, ZM10, JdM12]

Theories of Interest: Arrays

Used in software verification and hardware verification (for memories) [SBDL01, BNO+08a, dMB09]

Two interpreted function symbols read and write

Axiomatized by:

- $\forall a\, \forall i\, \forall v\ \mathrm{read}(\mathrm{write}(a, i, v), i) = v$
- $\forall a\, \forall i\, \forall j\, \forall v\ i \neq j \rightarrow \mathrm{read}(\mathrm{write}(a, i, v), j) = \mathrm{read}(a, j)$

Sometimes also with *extensionality*:

- $\forall a\, \forall b\ (\forall i\, \mathrm{read}(a, i) = \mathrm{read}(b, i)\ \rightarrow a = b)$

Is the following set of literals satisfiable in this theory?

$$\mathrm{write}(a, i, x) \neq b,\ \mathrm{read}(b, i) = y,\ \mathrm{read}(\mathrm{write}(b, i, x), j) = y,\ a = b,\ i = j$$

Theories of Interest: Bit vectors

Useful both in hardware and software verification [BCF+07, BB09, HBJ+14]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- *String-like*: concat, extract, ...
- *Logical*: bit-wise not, or, and, ...
- *Arithmetic*: add, subtract, multiply, ...
- *Comparison*: $<, >, \ldots$

Is this formula satisfiable over bit vectors of size 3?

$$a[1:0] \neq b[1:0] \;\wedge\; (a \mid b) = c \;\wedge\; c[0] = 0 \;\wedge\; a[1] + b[1] = 0$$

Implementing a Theory Solver: Difference Logic

We consider a simple example: difference logic.

In *difference logic*, we are interested in the satisfiability of a conjunction of arithmetic atoms.

Each atom is of the form $x - y \bowtie c$, where x and y are variables, c is a numeric constant, and $\bowtie \in \{=, <, \leq, >, \geq\}$.

The variables can range over either the *integers* (QF_IDL) or the *reals* (QF_RDL).

Difference Logic

The first step is to rewrite everything in terms of $\leq$:

Difference Logic

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- $x - y = c \quad \Longrightarrow \quad x - y \leq c \land x - y \geq c$

Difference Logic

The first step is to rewrite everything in terms of $\leq$:

- $x - y = c \quad \implies \quad x - y \leq c \wedge x - y \geq c$
- $x - y \geq c \quad \implies \quad y - x \leq -c$

Difference Logic

The first step is to rewrite everything in terms of $\leq$:

- $x - y = c \quad \implies \quad x - y \leq c \land x - y \geq c$
- $x - y \geq c \quad \implies \quad y - x \leq -c$
- $x - y > c \quad \implies \quad y - x < -c$

Difference Logic

The first step is to rewrite everything in terms of $\leq$:

- $x - y = c \quad \implies \quad x - y \leq c \wedge x - y \geq c$
- $x - y \geq c \quad \implies \quad y - x \leq -c$
- $x - y > c \quad \implies \quad y - x < -c$
- $x - y < c \quad \implies \quad x - y \leq c - 1$ (integers)

Difference Logic

The first step is to rewrite everything in terms of $\leq$:

- $x - y = c \quad \implies \quad x - y \leq c \wedge x - y \geq c$
- $x - y \geq c \quad \implies \quad y - x \leq -c$
- $x - y > c \quad \implies \quad y - x < -c$
- $x - y < c \quad \implies \quad x - y \leq c - 1$ (integers)
- $x - y < c \quad \implies \quad x - y \leq c - \delta$ (reals)

Difference Logic

Now we have a conjunction of literals, all of the form $x - y \leq c$.

From these literals, we form a weighted directed graph with a vertex for each variable.

For each literal $x - y \leq c$, there is an edge $x \xrightarrow{c} y$.

The set of literals is satisfiable iff there is no cycle for which the sum of the weights on the edges is negative.

There are a number of efficient algorithms for detecting negative cycles in graphs.

Difference Logic Example

$$x - y = 5 \ \wedge \ z - y \geq 2 \ \wedge \ z - x > 2 \ \wedge \ w - x = 2 \ \wedge \ z - w < 0$$

Difference Logic Example

$$x - y = 5 \;\wedge\; z - y \geq 2 \;\wedge\; z - x > 2 \;\wedge\; w - x = 2 \;\wedge\; z - w < 0$$

$$\begin{aligned}
&x - y = 5 \\
&z - y \geq 2 \\
&z - x > 2 \\
&w - x = 2 \\
&z - w < 0
\end{aligned}$$

Difference Logic Example

$$x - y = 5 \ \wedge \ z - y \geq 2 \ \wedge \ z - x > 2 \ \wedge \ w - x = 2 \ \wedge \ z - w < 0$$

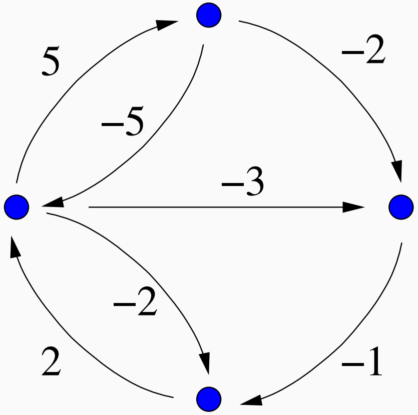
$$\begin{aligned} x - y &= 5 \\ z - y &\geq 2 \\ z - x &> 2 \qquad \Rightarrow \\ w - x &= 2 \\ z - w &< 0 \end{aligned}$$

Difference Logic Example

$$x - y = 5 \ \wedge \ z - y \geq 2 \ \wedge \ z - x > 2 \ \wedge \ w - x = 2 \ \wedge \ z - w < 0$$

$$\begin{array}{lcl} x - y = 5 & & x - y \leq 5 \wedge y - x \leq -5 \\ z - y \geq 2 & & y - z \leq -2 \\ z - x > 2 & \Rightarrow & x - z \leq -3 \\ w - x = 2 & & w - x \leq 2 \wedge x - w \leq -2 \\ z - w < 0 & & z - w \leq -1 \end{array}$$

Difference Logic Example



Suggested Readings

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